ON THE THEORY OF STABILITY OF MOTION

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This paper deals with an attempt to obtain a test for the stability of motion with the simultaneous use of several functions V. In this connection each function V can satisfy less rigid requirements than the one function occurring in the corresponding theorems of Liapunov's second method [1,2]. This allows us to expect that the use of several functions V can lead to a more flexible mechanism.

The work is based on Chaplygin's theory of differential inequalities [3]. That is, we shall apply the following theorem on differential inequalities of Wazewski [4].

Let the following system of equations be given

$$\frac{dy_s}{dt} = f_s(y_1, \ldots, y_k, t) \qquad (s = 1, \ldots, k) \tag{0.1}$$

where the f_s are definite and continuous in some open region Ω in a $(k+1)$-dimensional space; each function f_s is non-decreasing with respect to $y_1, \ldots, y_{s-1}, y_{s+1}, \ldots, y_k$ in region Ω. Then, through every interior point $(y_{10}, \ldots, y_{k0}, t)$ of region Ω there passes one upper integral $y^+(t, y_0, t_0)$ and one lower integral $y^-(t, y_0, t_0)$ of system (0.1) with respect to this point* and to the interval $[t_0, \alpha)$. The number α can be chosen equal to ∞ or such that as $t \to \alpha$ the representative point approaches the boundary of Ω along the upper (lower) integral.

Let functions $\psi_1(t), \ldots, \psi_k(t)$ be given, continuously differentiable in the interval $[t_0, \alpha)$, such that $\psi_s(t_0) = y_{s0}$, $(\psi_1(t), \ldots, \psi_k(t), t) \in \Omega$ when $t \in [t_0, \alpha)$.

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* These integrals are characterized by the fact that for every integral $y(t, y_0, t_0)$ passing through the point (y_0, t_0), for $t \in [t_0, \alpha)$:

$$y_s^-(t, y_0, t_0) \leqslant y_s(t, y_0, t_0) \leqslant y_s^+(t, y_0, t_0) \qquad (s = 1, \ldots, k)$$

1. If

$$\frac{d\psi_s(t)}{dt} \leqslant f_s(\psi_1(t), \ldots, \psi_k(t), t) \text{ when } t \in [t_0, \alpha) \qquad (s = 1, \ldots, k)$$

then

$$\psi_s(t) \leqslant y_s^+(t, y_0, t_0) \text{ when } t \in [t_0, \alpha) \qquad (s = 1, \ldots, k)$$

2. However, if

$$\frac{d\psi_s(t)}{dt} \geqslant f_s(\psi_1(t), \ldots, \psi_k(t), t) \text{ when } t \in [t_0, \alpha) \qquad (s = 1, \ldots, k)$$

then we shall have

$$\psi_s(t) \geqslant y_s^-(t, y_0, t_0) \text{ when } t \in [t_0, \alpha) \qquad (s = 1, \ldots, k)$$

It may be possible to apply other known theorems on differential and integral inequalities [5]. Then condition 3 in the obtained tests of stability and instability would be replaced by some other requirement.

The stability theorems obtained with the use of several functions V enable us to construct tests for stability and instability which utilize the properties of derivatives of the functions V of order higher than the first. We shall consider in detail such a family of tests with derivatives of the first and second order.

1. Let there be given the system of equations of perturbed motion

$$\frac{dx_i}{dt} = X_i(x_1, \ldots, x_n, t) \qquad (i = 1, \ldots, n) \tag{1.1}$$

The set of n real numbers $(x_1, \ldots, x_n)$ is considered as a point x in an n-dimensional space R^n with the norm $\|x\| = |x_1| + \ldots + |x_n|$.

The functions $X_i(x, t)$ are definite, continuous, and satisfy the Lipschitz conditions with respect to x in the region Γ

$$\|x\| \leqslant H, \qquad t \geqslant 0 \qquad (H = \text{const} > 0)$$

Let

$$X_i(0, t) \equiv 0 \qquad (i = 1, \ldots, n)$$

that is, system (1.1) admits of the unperturbed motion $x = 0$.

The perturbed motion is characterized by the set of functions

$$x(t, x_0, t_0) = \{x_1(t, x_{10}, \ldots, x_{n0}, t_0), \ldots, x_n(t, x_{10}, \ldots, x_{n0}, t_0)\}$$

which are definite and continuous when $(x_0, t_0) \in \Gamma$, $t \geqslant t_0$, and are continuously differentiable with respect to t.

Let us consider the real functions $V_1(x, t), \ldots, V_k(x, t)$ which are definite and continuous in region Γ together with their derivatives $\dot V_1(x, t), \ldots, \dot V_k(t)$ with respect to time t, taken relative to the equations of perturbed motion (1.1), and which vanish for the unperturbed motion, i.e. $V_s(0, t) \equiv 0$, $\dot V_s(0, t) \equiv 0$. For the set of these functions $V = (V_1, \ldots, V_k)$ we introduce the norm $\|V\| = |V_1| + \ldots + |V_k|$.

The functions $f_1(V, t), \ldots, f_k(V, t)$ will be assumed to be real, definite and continuous in region G

$$\|V\| < R_1, \quad t \geqslant 0 \qquad (R_1 > R = \sup [\|V(x, t)\| \text{when} (x, t) \in \Gamma] \quad \text{or} \quad R_1 = \infty)$$

Let us agree to call the functions $f_s(V, t)$ non-decreasing with respect to functions $V_1, \ldots, V_{s-1}, V_{s+1}, \ldots, V_k$ in G if, for arbitrary points

$$(V_1^*, \ldots, V_k^*, t^*) \in G, (V_1^{**}, \ldots, V_{s-1}^{**}, V_s^*, V_{s+1}^{**}, \ldots, V_k^{**} t^*) \in G$$

satisfying the inequalities

$$V_1^{**} \geqslant V_1^*, \ldots, V_{s-1}^{**} \geqslant V_{s-1}^*, V_{s+1}^{**} \geqslant V_{s+1}^*, \ldots, V_k^{**} \geqslant V_k^*$$

there holds

$$f_s(V_1^{**}, \ldots, V_{s-1}^{**}, V_s^*, V_{s+1}^{**}, \ldots, V_k^{**}, t^*) \geqslant f_s(V_1^*, \ldots, V_k^*, t^*)$$

For example, the function $f_s(V_s, t)$ not depending on $V_1, \ldots V_{s-1}$, $V_{s+1}, \ldots, V_k$ is non-decreasing with respect to $V_1, \ldots, V_{s-1}, V_{s+1}, \ldots, V_k$ in G.

Theorem 1.1. Let there exist functions $V_1(x, t), \ldots, V_k(x, t)$, possessing the following properties in Γ.

1. The functions $V_1(x, t) \geqslant 0, \ldots, V_l(x, t) \geqslant 0$ $(1 \leqslant l \leqslant k)$, and the function $V_1(x, t) + \ldots + V_l(x, t)$ is positive definite.

2. The derivatives relative to system (1.1) are

$$\dot V_s = f_s(V, t) + W_s(x, t) \qquad (s = 1, \ldots, k) \tag{1.2}$$

where $W_s(x, t) \leqslant 0$ and are continuous.

3. Each of the functions $f_s(V, t)$ is non-decreasing with respect to the functions $V_1, \ldots, V_{s-1}, V_{s+1}, \ldots, V_k$ in G.

4. The solution $y_1 = 0, \ldots, y_k = 0$ of the system

$$\frac{dy_s}{dt} = f_s(y_1, \ldots, y_k, t) \qquad (s = 1, \ldots, k) \tag{1.3}$$

is stable (or, asymptotically stable) with respect to $y_1, \ldots, y_l$ under the conditions $y_{10} \geqslant 0, \ldots, y_{l0} \geqslant 0$.

Then, the unperturbed motion $x = 0$ of system (1.1) is stable (or, asymptotically stable).

If the functions $V_1(x, t), \ldots, V_k(x, t)$ admit thereby of an infinitesimal upper bound and if the stability of the null solution of system (1.3) is uniform with respect to t_0 (or, the asymptotic stability is uniform with respect to $y_{10}, \ldots, y_{k0}, t_0$), then the stability of the unperturbed motion will be uniform with respect to t_0 (or, the asymptotic stability will be uniform with respect to x_0, t_0).

Proof. Let the conditions 1, 2, 3 be fulfilled and let the null solution of system (1.3) be stable with respect to $y_1, \ldots, y_l$ under the conditions $y_{10} \geqslant 0, \ldots, y_{l_0} \geqslant 0$.

Let there be given any positive number $A(0 < A < H)$. According to 1

$$0 < \inf [V_1(x, t) + \ldots + V_l(x, t) \text{ when } \|x\| \geqslant A, t \geqslant 0] \leqslant R$$

Therefore, if we take a positive number

$$\varepsilon(A) < \inf [V_1(x, t) + \ldots + V_l(x, t) \text{ when } \|x\| \geqslant A, t \geqslant 0]$$

then

$$\|x\| < A \text{ when } t \geqslant 0, \qquad V_1(x, t) + \ldots + V_e(x, t) \leqslant \varepsilon$$

By virtue of the assumption of stability for the null solution of system (1.3) with respect to $y_1, \ldots, y_l$ when $y_{10} \geqslant 0, \ldots, y_{l_0} \geqslant 0$, along $\varepsilon(A)$ for $t_0 \geqslant 0$, there is found a positive number $\delta(\varepsilon \ \ t_0)$ $(0 < \delta < \varepsilon < R)$ such that

$$|y_1^+(t, y_0, t_0)| + \ldots + |y_l^+(t, y_0, t_0)| < \varepsilon$$

for all $t \geqslant t_0$ when $|y_{10}| + \ldots + |y_{k0}| \leqslant \delta$, $y_{10} \geqslant 0, \ldots, y_{l0} \geqslant 0$ (the upper integral $y^+(t, y_0, t_0)$ of system (1.3) exists according to Wazewski's theorem).

The function $|V_1(x, t_0)| + \ldots + |V_k(x, t_0)|$ admits of an infinitesimal upper bound, and therefore for δ and t_0 there is found a positive number $\eta(\delta, t_0) = \eta(A, t_0)$ such that

$$|V_1(x_0, t_0)| + \ldots + |V_k(x_0, t_0)| \leqslant \delta \quad \text{when} \|x_0\| \leqslant \eta$$

Let us show that for any perturbed motion $x(t, x_0, t_0)$

$$\|x(t, x_0, t_0)\| < A \quad \text{when } t \geqslant t_0$$

and when the initial data is $\|x_0\| \leqslant \eta$, $t_0 \geqslant 0$ $(0 < \eta(A, t_0) < A)$.

Let us assume that this is not the case, i.e. there are found x_0^*, t^* $(\|x_0^*\| \leqslant \eta$, $t^* > t_0)$ such that $\|x(t, x_0^*, t_0)\| < A$ when $t \in [t_0, t^*)$, but $\|x(t^*, x_0^*, t_0)\| = A$.

Let us set $y_{s0}^* = V_s(x_0^*, t_0)$. Then by choosing η

$$|y_{10}^*| + \ldots + |y_{k0}^*| = |V_1(x_0^*, t_0^*)| + \ldots + |V_k(x_0^*; t_0)| \leqslant \delta$$

but according to 1

$$y_{10}^* \geqslant 0, \ldots, y_{l0}^* \geqslant 0$$

and by choosing δ

$$|y_1^+(t, y_0^*, t_0)| + \ldots + |y_l^+(t, y_0^*, t_0)| < \varepsilon \text{ on } [t_0, t^*]$$

Let us consider (as the solution of system (1.1), (1.2) with continuous right-hand sides) the functions $V_s(x(t, x_0^*, t_0), t)$ which are continuously differentiable with respect to t in the interval $[t_0, t^* + \Delta t)$. By virtue of 2

$$\frac{dV_s(x(t, x_0^*, t_0), t)}{dt} \leqslant f_s(V(x(t, x_0^*, t_0), t), t) \qquad (s = 1, \ldots, k)$$

when $t \in [t_0, t^* + \Delta t)$, $(\Delta t > 0$ is sufficiently small), therefore, by applying Wazewski's theorem we get

$$V_s^-(x(t, x_0^*, t_0), t) \leqslant y_s^+(t, y_0^*, t_0) \qquad (s = 1, \ldots, k)$$

when $t \in [t_0, t^*]$, and consequently

$$\sum_{s=1}^{l} |V_s(x(t, x_0^*, t_0), t) \leqslant \sum_{s=1}^{l} |y_s^+(t, y_0^*, t_0)| < \varepsilon$$

But then, by choosing ε, $\|x(t, x_0^*, t_0)\| < A$ for $t \in [t_0, t^*]$ and, in particular, $\|x(t^*, x_0^*, t_0)\| < A$, which contradicts the assumption we have made. The contradiction proves the stability of the unperturbed motion $x = 0$ of system (1.1).

Here, if the stability of the null solution of system (1.3) is uniform with respect to t_0 and if the functions $V_1, \ldots, V_k$ admit of an infinitesimal upper bound, then the numbers $\delta(\varepsilon)$ and $\eta(\delta) = \eta(A)$ may be chosen independently of t_0, i.e. the stability of the unperturbed motion $x = 0$ of system (1.1) will be uniform with respect to t_0.

Let the null solution of system (1.3) be asymptotically stable with respect to $y_1, \ldots, y_l$ under the conditions $y_{10} \geqslant 0, \ldots, y_{l0} \geqslant 0$, i.e. along with any positive number $\alpha < \delta$ for given t_0, y_0, ($\|x_0\| \leqslant \eta$, $|y_{10}| + \ldots + |y_{k0}| \leqslant \delta$, $y_{10} \geqslant 0, \ldots, y_{l0} \geqslant 0$), there if found a $T(\alpha, t_0, y_0) > 0$ such that

$$\sum_{s=1}^{l} |y_s^+(t, y_0, t_0)| < \alpha \quad \text{when } t > t_0 + T$$

Then

$$\sum_{s=1}^{l} V_s(x(t, x_0, t_0)\,t) < \alpha \text{ when } t > t_0 + T$$

In fact, by assuming contrarily the existence of a $t^+ \in (t_0 + T, \infty)$ such that $V_1(x(t^+, x_0, t_0), t^+) + \ldots + V_l(x(t^+, x_0, t_0), t^+) \geqslant \alpha$, we are led to a contradiction with the estimate

$$\sum_{s=1}^{l} V_s(x(t, x_0, t_0), t) \leqslant \sum_{s=1}^{l} y_s^+(t, y_0, t_0)$$

which can be derived for the segment $[t_0, t^+]$ analogously to the previous case.

Thus, for $\|x_0\| \leqslant \eta$ we have

$$\lim_{t\to\infty} \sum_{s=1}^{l} V_s(x(t, x_0, t_0), t) = 0$$

By virtue of the positive definiteness of $V_1(x, t) + \ldots + V_l(x, t)$ it follows that $\lim \|x(t, x_0, t_0)\| = 0$ as $t \to \infty$, and that the unperturbed motion $x = 0$ of system (1.1) is asymptotically stable.

Here if the asymptotic stability of the null solution of system (1.3) is uniform with respect to y_0, t_0, and if the functions $V_1, \ldots, V_k$ admit of an infinitesimal upper bound, then the number T can be chosen independently of t_0, y_0, x_0, i.e.

$$\sum_{s=1}^{l} V_s(x(t, x_0, t_0), t) \to 0 \text{ when } t \to \infty$$

uniformly with respect to x_0, t_0. Hence it is easily concluded that when $t \to \infty$, $\|x(t, x_0, t_0)\| \to 0$ uniformly with respect to x_0, t_0, and then from this it follows that the asymptotic stability of the unperturbed motion $x = 0$ of system (1.1) is uniform with respect to x_0, t_0. The theorem is proved.

Corollary ($k = l = 1$). In Γ let there exist a positive definite function $V(x, t)$ whose derivative relative to system (1.1) is

$$\dot{V} = f(V, t) + W(x, t)$$

where $W(x, t) \leqslant 0$ and $f(V, t)$ is such that the solution $y = 0$ of the equation

$$\frac{dy}{dt} = f(y, t) \qquad (1.4)$$

is stable (or, asymptotically stable) when $y_0 \geqslant 0$.

Then the unperturbed motion $x = 0$ of system (1.1) is stable (or, asymptotically stable). Here if the function V admits of an infinitesimal upper bound and if the stability of the null solution of equation (1.4) is uniform with respect to t_0 (or, the asymptotic stability is uniform with respect to x_0, t_0), then the stability of the unperturbed motion will be uniform with respect to t_0 (or, the asymptotic stability will be uniform with respect to x_0, t_0).

This proposition has been proved by Korduniany [6], and in turn, it generalizes the classical theorem of Liapunov [1] on the stability of motion

$$f(V, t) \equiv 0$$

its modification, proposed by Ibrashev [7]

$$f = L\,|\,\theta(t)\,|\,V, \qquad L = \text{const} > 0, \qquad \int_0^\infty |\,\theta(t)\,|\,dt < \infty$$

Persidskii's theorem [8] on uniform stability

$$f \equiv 0$$

Liapunov's theorem on asymptotic stability [1] and its modifications obtained by Massera [9], Krasovskii [10], Zubov [11]

$$f = -\varphi(t)\,c(V) \ (\varphi(t) \geqslant 0, \int_0^\infty \varphi\ dt = \infty,\ c(0) = 0,$$

(c is a strongly increasing function of V);

Malkin's theorem [12] on uniform asymptotic stability [9]

$$f = -c(V)$$

It is also an adjunct of the results of Stokes [13] and Rakhmatullina [14].

Example 1.1. Problem of the stability in the sense of Liapunov of bodies with variable mass.

Let the right-hand sides of the equations of perturbed motion (1.1) be holomorphic functions of x with continuously differentiable and bounded coefficients, approaching constants as $t \to \infty$, i.e. $\lim X_i(x, t) = X_i^*(x)$ as $t \to \infty$.

In the case of the limit system

$$\frac{dx_i}{dt} = X_i^*(x) \qquad (i = 1, \ldots, n)$$

which according to Liapunov is a special case, Aminov [15] has proposed a method of constructing the functions V. The function of Aminov is a quadratic form

$$V = \frac{1}{2} \sum_{i,j=1}^{n} p_{ij}(t) x_i x_j \qquad (p_{ij}(t) = p_{ji}(t))$$

whose derivative relative to the equations of perturbed motion (1.1) is

$$\dot{V} = \sum_{i,j=1}^{n} \frac{dp_{ij}}{dt} x_i x_j$$

The requirements of positive-definiteness of the function V and of the negative semidefiniteness of the derivative $\dot{V}$ are sufficient conditions for stability according to Liapunov's theorem. For bodies of variable mass they do not usually coincide with the necessary conditions for stability [15]. Let us see how these sufficient conditions for stability can be relaxed when the Aminov's functions are used in Kordunianu's theorem. By assuming that V is positive definite, we find a positive number B such that

$$V \geqslant \frac{1}{B} \sum_{\nu=1}^{n} x_\nu^2$$

Let us transform the derivative $\dot{V}$ to the following form

$$\dot{V} = \frac{1}{2} \sum_{i,j=1}^{n} \left| \frac{dp_{ij}}{dt} \right| (x_i^2 + x_j^2) - \frac{1}{2} \sum_{i,j=1}^{n} \left| \frac{dp_{ij}}{dt} \right| (x_i \pm x_j)^2 \leqslant$$

$$\leqslant \frac{1}{2} \sum_{i,j=1}^{n} \left| \frac{dp_{ij}}{dt} \right| (x_i^2 + x_j^2) \leqslant \sum_{i,j=1}^{n} \left| \frac{dp_{ij}}{dt} \right| \sum_{\nu=1}^{n} x_\nu^2 \leqslant BV \sum_{i,j=1}^{n} \left| \frac{dp_{ij}}{dt} \right|$$

If

$$\int_{t_0}^{\infty}\left|\frac{dp_{ij}}{dt}\right|dt<\infty \qquad (i, j=1,\ldots,n) \tag{1.5}$$

then the solution $y = 0$ of the equation

$$\frac{dy}{dt}=B\sum_{i,j=1}^{n}\left|\frac{dp_{ij}}{dt}\right|y$$

is stable, and the conditions of Kordunianu's theorem (as well as the conditions of Ibrashev's theorem) are fulfilled since here the unperturbed motion $x = 0$ is stable.

However, for bodies of variable mass

$$\left|\int_{t_0}^{\infty}\frac{dp_{ij}}{dt}dt\right|=|p_{ij}(\infty)-p_{ij}(t_0)|<\infty$$

and therefore (1.5) holds, if the derivatives dp_{ij}/dt change sign a finite number of times on the semi-axis $[0, \infty)$. As is apparent, this happens in a large number of practical cases of interest. In these cases, therefore, the only sufficient stability condition is the condition of positive definiteness of quadratic form V which, as from the results of Aminov [15], often is also the necessary stability condition.

Example 1.2.

$$\begin{aligned}\frac{dx_1}{dt}&=(\sin t+e^{-t})x_1+(\sin t-e^{-t})x_2-\sin^2 t\,(x_1^3+x_1x_2^2)\\ \frac{dx_2}{dt}&=(\sin t-e^{-t})x_1+(\sin t+e^{-t})x_2-\sin^2 t\,(x_1^2x_2+x_2^3)\end{aligned} \tag{1.6}$$

Let us seek the Liapunov function as a quadratic form with constant coefficients

$$V={}^1/_2\,(x_1^2+2Bx_1x_2+Ax_2^2)$$

Its derivative relative to system (1.6) is

$$\dot V=\dot V^{(2)}+\dot V^{(4)}$$

$$V^{(2)}=[(A+B)\sin t+(A-B)e^{-t}]\,x_1^2+[(1+A+2B)\sin t+(2B-A-1)e^{-t}]\,x_1x_2+ \\ +[(A+B)\sin t+(A-B)e^{-t}]\,x_2^2$$

$$\dot V^{(4)}=-\sin^2 t\,[x_1^4+2Bx_1^3x_2+(1+A)x_1^2x_2^2+2Bx_1x_2^3+Ax_2^4]$$

For arbitrary A and B the function V does not satisfy Liapunov's theorem on the stability of motion. Let us try to satisfy Kordunianu's theorem by assuming $\dot V^{(2)} = \varphi(t)V$.

This equality can occur in two cases:

1) $A_1 = B_1 = 1, \qquad \varphi_1(t) = 4\sin t \quad$ when $V_1 = {}^1/_2 (x_1 + x_2)^2$

2) $A_2 = 1, \quad B_2 = -1, \quad \varphi_2(t) = 4e^{-t} \quad$ when $V_2 = {}^1/_2 (x_1 - x_2)^2$

The function V_1, and also V_2, will not be positive definite functions and, consequently, will not satisfy Kordunianu's theorem. However, the two functions V_1 and V_2 satisfy the conditions of Theorem 1.1. In fact,

1. The functions $V_1 \geqslant 0$, $V_2 \geqslant 0$ admit of an infinitesimal upper bound, and the function $V_1 + V_2 = x_1^2 + x_2^2$ is positive definite.

2. The derivatives are $\dot{V}_1 \leqslant 4 \sin t\, V_1$, $\dot{V}_2 \leqslant 4\, e^{-t}\, V_2$.

3. The function $4 \sin t\, V_1$ does not decrease with respect to V_2, and the function $4\, e^{-t}\, V_2$ does not decrease with respect to V_1.

4. The null solution of the equation

$$\frac{dy_1}{dt} = 4\sin t y_1, \qquad \frac{dy_2}{dt} = 4e^{-t}\, y_2$$

is uniformly stable with respect to t_0.

Hence the unperturbed motion $x_1 = 0$, $x_2 = 0$ of system (1.6) is uniformly stable with respect to t_0.

2. Let the functions $f_s(V, t)$ be definite and continuous in G or in the half-space $E(t \geqslant 0)$ of the $(k + 1)$-dimensional space $\{V, t\}$.

Definition. The null solution of system (1.3) is called $+y_1$-unstable (or, $+y_1$-unstable in G) if for any positive numbers δ, ε, t_0, satisfying the conditions $0 < \delta < \varepsilon < R$ and ε sufficiently small (or, $\delta < \varepsilon = R$, or $\delta < \varepsilon < \infty$ when ε is arbitrarily large), there is found a positive number T and a point $x_0(\|x_0\| \leqslant \delta)$ such that every solution $y(t, y_0, t_0)$ of system (1.3) with initial data $y_{s0} = V_s(x_0, t_0)$ $(s = 1, \ldots, k)$, $t_0 \geqslant 0$, for all values of $t \in [t_0, t_0 + T]$, remains in G and satisfies the conditions

$$y_1(t_0 + T, y_{10}, \ldots, y_{k0}, t_0) > \varepsilon, \quad |y_{10}| + \ldots + |y_{k0}| \leqslant \delta$$

For example, the null solution of the equation

$$\frac{dy_1}{dt} = \varphi(y_1)\, p(t) \qquad \left(p(t) \geqslant 0, \ \int_{t_0}^{\infty} p(t)\, dt = \infty\right) \tag{2.1}$$

where $\varphi(y_1) > 0$ when $y_1 > 0$ and $\varphi(0) = 0$, is $+y_1$-unstable in the half-plane $E(t \geqslant 0)$ if the function $V_1(x, t)$ can take positive values for arbitrarily small $\|x\|$ and for any $t > 0$.

Theorem 2.1. Let there exist functions $V_1(x, t), \ldots, V_k(x, t)$ having the following properties in Γ.

1. The function $V_1(x, t)$ admits of an infinitesimal upper bound, (or, is bounded).

2. The derivatives relative to system (1.1) are

$$\dot{V}_s = f_s(V, t) + W_s(x, t) \qquad (s = 1, \ldots, k) \tag{2.2}$$

where $W_s(x, t) \geqslant 0$ and are continuous.

3. Each of the functions $f_s(V, t)$ will be non-decreasing with respect to the functions $V_1, \ldots, V_{s-1}, V_{s+1}, \ldots, V_k$ in region G.

4. The null solution of the system

$$\frac{dy_s}{dt} = f_s(y_1, \ldots, y_k, t) \qquad (s = 1, \ldots, k) \tag{2.3}$$

is $+y_1$-unstable (or, $+y_1$-unstable in G).

Then the unperturbed motion $x = 0$ of system (1.1) is unstable.

Proof. Let the conditions of the theorem be fulfilled. According to 1, for arbitrarily small $0 < \varepsilon < R$ (or, for $\varepsilon = R$, or for sufficiently large $\varepsilon > 0$) we can find an $h(0 < h < H)$ such that $V_1(x, t) \leqslant \varepsilon$ when $\|x\| \leqslant h$, $t \geqslant 0$. It is then required to prove that for an arbitrary number $A(0 < A < h)$ and for $t_0 \geqslant 0$ there cannot be found a $\lambda(0 < \lambda < A)$ such that when $\|x\| \leqslant \lambda$ for all $t \geqslant t_0$ we would have $\|x(t, x_0, t_0)\| < A$.

Let us assume, contrarily, that such a λ does exist. Let us designate $y_{s0} = V_s(x_0, t_0)$. By virtue of the continuity of V_s with respect to x_0 we can assume that λ is so small that

$$0 < \sum_{s=1}^{k} |y_{s0}| = \sum_{s=1}^{k} |V_s(x_0, t_0)| < \varepsilon$$

According to 4, there can be found a $T > 0$ and $\|x_0^*\| \leqslant \lambda$ such that for all $t \in [t_0, t_0 + T]$ the solutions $y(t, y_0^*, t_0)$ of system (2.3) will remain in G, and $y_1(t_0 + T, y_0^*, t_0) > \varepsilon$.

Moreover, by assumption, for all $t \geqslant 0$ we shall have

$$\sum_{s=1}^{k} |V_s(x(t, x_0^*, t_0), t)| \leqslant R \ \left(\text{or} \ \sum_{s=1}^{k} |V_s(x(t, x_0^*, t_0), t)| < \infty\right)$$

The functions $V_s(x(t, x_0^*, t_0), t)$ are continuously differentiable in the interval $[t_0, t_0 + T + \Delta t)$, and by virtue of 2 in this interval they satisfy the inequalities

$$\frac{dV_s(x(t, x_0^*, t_0), t)}{dt} \geqslant f_s(V(x(t, x_0^*, t_0)t), t) \qquad (s = 1, \ldots, k)$$

when $\Delta t > 0$ is sufficiently small. Hence, by virtue of 3 Wazewski's theorem is also applicable, according to which there exists the lower integral $y^-(t, y_0^*, t_0)$ and

$$V_s(x(t, x_0^*, t_0), t) \geqslant y_s^-(t, y_0^*, t_0) \qquad \text{for } t \in [t_0, t_0 + T] \qquad (s = 1, \ldots, k)$$

and, in particular, $V_1(x(t, x_0^*, t_0), t) \geqslant y_1^-(t, y_0^*, t_0)$.

But then $V_1(x(t_0 + T; x_0^*, t_0), t_0 + T) \geqslant y_1^-(t_0 + T, y_0^*, t_0) > \varepsilon$, which, according to the choice of ε, signifies that $\| x(t_0 + T, x_0^*, t_0)\| > h > A$ in contradiction to the assumption we made. The contradiction proves the theorem.

Corollary ($k = 1$). Let there exist a function $V(x, t)$, admitting of an infinitesimal upper bound (or, being bounded), which can take positive values for arbitrarily small $\|x\|$ and for any $t > 0$, and whose derivative relative to system (1.1) is $\dot{V} \geqslant f(V, t)$ where $f(V, t) \geqslant 0$ when $t \geqslant 0$ and $0 < V < \sup[V \text{ when } (x, t) \in \Gamma]$ (or, for any $0 < V < \infty$) and such that for an arbitrary positive number l there is found a continuous function

$$m(t) \geqslant 0, \qquad \int_0^\infty m(t)\,dt = \infty$$

such that

$$f(V, t) \geqslant m(t) \text{ when } t \geqslant 0, \qquad l < V < \sup[V \text{ when } (x, t) \in \Gamma]$$

(respectively, for any $l < V < \infty$). Then the perturbed motion $x = 0$ of system (1.1) is unstable.

This proposition is a modification of Chetaev's instability theorem [2] and contains both the classical theorems of Liapunov [1]

$$f = m(t)\,\varphi(V), \qquad \varphi(V) > 0 \text{ when } V > 0$$

and certain of their generalizations [9, 11].

3. Let there be given a real function $V(x, t)$ continuous in Γ, having in Γ continuous partial derivatives up to the kth order with respect to $x_1, \ldots, x_n, t$. Let the functions $X_1, \ldots, X_n$ have continuous derivatives up to the $(k - 1)$st order in Γ. Let us denote the derivative $\dot{V}$ relative

to system (1.1) of the function V by $V^{(1)}(x, t)$

$$V^{(1)} = \dot{V} = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} X_i(x, t) + \frac{\partial V}{dt}$$

The second derivative of the function V relative to system (1.1) is given by

$$V^{(2)}(x, \dot{t}) = \sum_{i=1}^{n} \frac{\partial V^{(1)}}{\partial x_i} X_i(x, t) + \frac{\partial V^{(1)}}{dt}$$

If the derivatives $V^{(1)}, V^{(2)}, \ldots, V^{(s)}$ are thus defined, then the derivative of the $(s + 1)$st order of the function V relative to system (1.1) is given by

$$V^{(s+1)}(x, t) = \sum_{i=1}^{n} \frac{\partial V^{(s)}(x, t)}{\partial x_i} X_i(x, t) + \frac{\partial V^{(s)}(x, t)}{\partial t} \qquad (s + 1 \leqslant k)$$

From Theorem 1.1 ensues the following test for the stability of motion.

Let there exist a positive definite function $V(x, t)$ whose kth order derivative relative to system (1.1) satisfies the condition $V^{(k)} \leqslant f(V, V^{(1)}, \ldots, V^{(k-1)}, t)$, where the function f is non-decreasing with respect to $V, V^{(1)}, \ldots, V^{(k-2)}$, and let the null solution of the equation

$$\frac{d^k y}{dt^k} = f\left(y, \frac{dy}{dt}, \ldots, \frac{d^{k-1}y}{dt^{k-1}}, t\right)$$

be stable (or, asymptotically stable) with respect to y when $y_0 \geqslant 0$. Then the unperturbed motion $x = 0$ of system (1.1) is stable (or, asymptotically stable).

In fact, the functions $V_1 = V$, $V_2 = V^{(1)}, \ldots, V_k = V^{(k-1)}$ satisfy the conditions of Theorem 1.1 for $l = 1$, since $f_s = V^{(s)}$ $(s = 1, \ldots, k - 1)$ does not decrease with respect to $V, V^{(1)}, \ldots, V^{(k-1)}$, and $af_k = f$ does not decrease with respect to $V, V^{(1)}, \ldots, V^{(k-2)}$.

From Theorem 2.1 a test for instability is obtained in an analogous manner.

Let there exist a function $V(x, t)$, admitting of an infinitesimal upper bound (or, being bounded), whose kth order derivative relative to system (1.1) $V^{(k)} \geqslant f(V, V^{(1)}, \ldots, V^{(k-1)}, t)$, where the function f is non-decreasing with respect to $V, V^{(1)}, \ldots, V^{(k-2)}$, and let the null solution of the equation

$$\frac{d^k y}{dt^k} = f\left(y, \frac{dy}{dt}, \ldots, \frac{d^{k-1}y}{dt^{k-1}}, t\right)$$

be +y-unstable in the region $|y| + |dy/dt| + \ldots + |d^{k-1}y/dt^{k-1}| \leqslant R$, $t \geqslant 0$, or in the half-space $E(t \geqslant 0)$ of the $(k + 1)$-dimensional space $\{y, dy/dt, \ldots, d^{k-1}y/dt^{k-1}, t\}$. Then the unperturbed motion $x = 0$ of system (1.1) is unstable.

Let us consider in detail the case of $k = 2$ and, which is of most interest in applications of a linear function, f.

Theorem 3.1. If there exists a positive definite function $V(x, t)$ whose second derivative relative to system (1.1) $V^{(2)} \leqslant p(t)V^{(1)}$, where the continuous function $p(t)$ satisfies the condition

$$\int_{t_0}^{\infty} \exp \int_{t_0}^{t} p(\tau)\, d\tau\, dt < \infty \tag{3.1}$$

then the unperturbed motion $x = 0$ of system (1.1) is stable.

Proof. Under the conditions of the theorem the function $f = p(t)V^{(1)}$ is non-decreasing with respect to V since, clearly, it does not contain V. The null solution of the equation

$$\frac{d^2y}{dt^2} = p(t)\frac{dy}{dt}$$

is stable with respect to y; this follows from the form of the general solution of this equation

$$y(t) = y_0 + \left(\frac{dy}{dt}\right)_0 \int_{t_0}^{t} \exp \int_{t_0}^{\tau} p(\nu)\, d\nu\, d\tau$$

and from the boundary condition occurring in its integral. Therefore, the conditions of the stability test, which were formulated at the beginning of this paragraph and on the basis of which we decided upon the stability of the motion, are satisfied.

Condition (3.1) is satisfied, for example, by the functions

$$p = \text{const} < 0, \qquad p = -\frac{a}{t} \qquad (a = \text{const} > 1)$$

Theorem 3.2. If there exists a bounded function $V(x, t)$ whose second derivative relative to system (1.1) satisfies the condition $V^{(2)} \geqslant aV + 2bV^{(1)}$, where a and b are constants, $a \geqslant 0$ and, moreover, $b \geqslant 0$ when $a = 0$, and if the function $[\sqrt{(b^2 + a)} - b]V(x, t) + V^{(1)}(x, t)$ can take positive values for arbitrarily small $\|x\|$ and any $t > 0$, then the unperturbed motion of system (1.1) is unstable.

Proof. Under the conditions of the theorem the function $f = aV + 2bV^{(1)}$ is non-decreasing with respect to V since $a \geqslant 0$. The null solution of the equation

$$\frac{d^2y}{dt^2} = ay + 2b\frac{dy}{dt}$$

is +y-unstable in the half-space $E(t \geqslant 0)$ of the three-dimensional space $\{y,\ dy/dt,\ t\}$. In fact, the general solution of this equation in the case $a > 0$ is

$$y = \frac{(\sqrt{b^2+a} - b)\,y_0 + (dy/dt)_0}{2\sqrt{b^2+a}}\,e^{(\sqrt{b^2+a}+b)(t-t_0)} + \\ + \frac{(\sqrt{b^2+a}+b)\,y_0 - (dy/dt)_0}{2\sqrt{b^2+a}}\,e^{(b-\sqrt{b^2+a})(t-t_0)}$$

According to the conditions of the theorem there is found an arbitrarily small $\|x_0\|$ such that

$$(\sqrt{b^2+a} - b)\,V(x_0, t_0) + V^{(1)}(x_0, t_0) = (\sqrt{b^2+a} - b)\,y_0 + (dy/dt)_0 > 0$$

and, consequently, $y \to \infty$ as $t \to \infty$. However, if $a = 0$, then the general solution has the form

$$y = y_0 + \frac{(dy/dt)_0}{2b}\,[e^{2b(t-t_0)} - 1] \quad (b > 0), \qquad y = y_0 + \left(\frac{dy}{dt}\right)_0 (t - t_0) \quad (b = 0)$$

and here also $y \to \infty$ as $t \to \infty$,

$$(dy/dt)_0 = V^{(1)}(x_0, t_0) > 0$$

Since the conditions of the instability test formulated above are fulfilled, the theorem is proved.

The second derivative of function V relative to the equations of perturbed motion, was used by Ibrashev who proposed a theorem on the instability of motion [7]. We can prove the following extension of Ibrashev's theorem.

Theorem 3.3. If there exists a bounded function $V(x,\ t)$ such that for arbitrarily small $\|x\|$ and any $t \geqslant 0$ the functions $V(x,\ t)$ and $V^{(1)}(x, t)$ simultaneously may take positive values, and $V^{(2)}(x,\ t) \geqslant 0$ $(V^{(2)} \equiv 0)$ in the set $E(V > 0,\ V^{(1)} > 0)$, then the unperturbed motion $x = 0$ of system (1.1) is unstable.

Let us choose, in an arbitrarily small neighborhood of the unperturbed motion, a point $(x_0,\ t_0) \in \Gamma$ at which

$$V(x_0, t_0) > 0, \qquad V^{(1)}(x_0, t_0) > 0$$

The perturbed motion $x(t, x_0, t_0)$ will remain in the set $E(V > 0, V^{(1)} > 0)$ until it leaves Γ since otherwise for some $T > t_0$ we would have $V(x(T, x_0, t_0), T) V^{(1)}(x(T, x_0, t_0), T) = 0$ when $V(x(t, x_0, t_0), t) V^{(1)}(x(t, x_0, t_0), t) > 0$ for $t \in [t_0, T)$.

But this is impossible since for $t \in [t_0, T)$

$$V(x(t, x_0, t_0), t) \cdot V^{(1)}(x(t, x_0, t_0), t) =$$
$$= [V(x_0, t_0) + \int_{t_0}^{t} V^{(1)} dt] [V^{(1)}(x_0, t_0) + \int_{t_0}^{t} V^{(2)} dt] \geqslant V(x_0, t_0) V^{(1)}(x_0, t_0)$$

Consequently (by virtue of the continuity of $V(x(t, x_0, t_0), t) V^{(1)}(x(t, x_0, t_0), t)$ with respect to t)

$$V(x(T, x_0, t_0), T)\ V^{(1)}(x(T, x_0, t_0), T) \geqslant V(x_0, t_0)\ V^{(1)}(x_0, t_0) > 0$$

But in the set $E(V > 0, V^{(1)} > 0)$

$$V^{(1)}(x(t, x_0, t_0), t) = V^{(1)}(x_0, t_0) + \int_{t_0}^{t} V^{(2)} dt \geqslant V^{(1)}(x_0, t_0)$$

$$V(x(t, x_0, t_0), t) = V(x_0, t_0) + \int_{t_0}^{t} V^{(1)} dt \geqslant V(x_0, t_0) + V^{(1)}(x_0, t_0)(t - t_0)$$

The incompatibility of the latter inequality with the condition of boundedness of $V(x, t)$ indicates the instability of the motion.

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